

# VOLUME GROWTH OF QUASIHYPHERBOLIC BALLS

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**ABSTRACT.** The purpose of this paper is to discuss the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain  $G \subset \mathbb{R}^n$ , in terms of the radius. It turns out that in the case of domains with Ahlfors regular boundaries, the rate of growth depends not merely on the radius but also on the metric structure of the boundary.

**KEYWORDS.** Quasihyperbolic volume, uniform porosity,  $Q$ -regularity

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## 1. INTRODUCTION

Since its introduction three decades ago, the quasihyperbolic metric has become a popular tool in many subfields of geometric function theory. For instance, in the study of quasiconformal maps of  $\mathbb{R}^n$  and of Banach spaces [Va1], analysis of metric spaces [H] and hyperbolic type metric [HIMPS]. A natural question is whether and to what extent, the results of hyperbolic geometry have counterparts for the quasihyperbolic geometry. For instance in [K] it was noticed that some facts from hyperbolic trigonometry of the plane have counterparts in the quasihyperbolic setup while some have not.

The purpose of this paper is to discuss the notion of the quasihyperbolic volume and to find growth estimates for the quasihyperbolic volume of balls in a domain  $G \subset \mathbb{R}^n$ , in terms of the radius. It turns out that the rate of growth depends not merely on the radius but also on the metric structure of the boundary. Below we give an explicit growth estimate for the case of domains with Ahlfors regular boundary.

For a compact set  $E \subset \mathbb{R}^n$ , and  $0 < s < t$ , we consider the layer sets  $E(s, t) = \{z : s \leq d(z, E) \leq t\}$  and relate its volume to the metric size of  $E$  via its (Hausdorff) dimension under the additional assumption that the boundary of  $E$  be Ahlfors  $Q$ -regular for some  $0 < Q < n$ . It is practically equivalent to formulate

this idea in terms of the number of those Whitney cubes of the Whitney decomposition of  $\mathbb{R}^n \setminus E$  that meet  $E(s, t)$ . This idea goes back to [MV]—in Lemmas 3.1 and 3.3 we will refine some results of [MV]. One of the key ideas of [MV] was to estimate from above the metric size of  $E$  in terms of the size of the layer sets.

In the last section, we estimate from below the growth of the quasihyperbolic volume of balls in the case when  $G = \mathbb{R}^n \setminus E$  is  $\psi$ -uniform and  $E$  is Ahlfors regular. The main result of the paper is Theorem 4.19 which gives a lower bound for the volume growth.

## 2. NOTATION

The *quasihyperbolic distance* between two points  $x$  and  $y$  in a proper subdomain  $G$  of the Euclidean space  $\mathbb{R}^n$ ,  $n \geq 2$ , is defined by

$$k_G(x, y) = \inf_{\alpha \in \Gamma_{xy}} \int_{\alpha} \frac{|dz|}{d(z)},$$

where  $d(z) = d(z, \partial G)$  is the (Euclidean) distance between the point  $z \in G$  and the boundary of  $G$  (denoted  $\partial G$ ), and  $\Gamma_{xy}$  is the collection of all rectifiable curves in  $G$  joining  $x$  and  $y$ . Some basic properties of the quasihyperbolic metric can be found in [V1].

In particular, G.J. Martin and B.G. Osgood [MO, page 38] showed that for  $x, y \in \mathbb{R}^n \setminus \{0\}$  and  $n \geq 2$

$$(2.1) \quad k_{\mathbb{R}^n \setminus \{0\}}(x, y) = \sqrt{\theta^2 + \log^2 \frac{|x|}{|y|}},$$

where  $\theta = \angle(x, 0, y) \in [0, \pi]$ .

For the volume of the unit ball in  $\mathbb{R}^n$  we use the notation  $\Omega_n = m(B(0, 1)) = \pi^{n/2}/\Gamma((n/2) + 1)$  where  $m$  is the  $n$ -dimensional Lebesgue measure and  $\Gamma$  stands for the usual  $\Gamma$ -function, see [AS, Ch 6]. The surface area of the unit sphere is  $\omega_{n-1} = m_{n-1}(S^{n-1}(0, 1)) = n\Omega_n$ .

The quasihyperbolic volume of a Lebesgue measurable set  $A \subset G$  is defined by

$$\text{vol}_{k_G}(A) = \int_A \frac{dm(z)}{d(z)^n}.$$

We also use  $\text{vol}_k(A)$  for  $\text{vol}_{k_G}(A)$  if the domain  $G$  is clear from the context.

We assume  $n \geq 2$  and use the notation  $B^n(x, r) = \{z \in \mathbb{R}^n : |x - z| < r\}$  for the Euclidean ball, and its boundary is the sphere  $S^{n-1}(x, r) = \partial B^n(x, r)$ , where the center  $x$  can be omitted if  $x = 0$ .

Below we use the Whitney decomposition of the complement of a closed set in  $\mathbb{R}^n$ . If  $E \subset \mathbb{R}^n$  is non-empty and closed, then  $\mathbb{R}^n \setminus E$  can be presented [S, p. 16] as a union of closed dyadic cubes  $Q_j^k$

$$(2.2) \quad \mathbb{R}^n \setminus E = \bigcup_{k \in \mathbb{Z}} \bigcup_{j=1}^{N_k} Q_j^k,$$

where the cubes  $Q_j^k$  have the following properties:

- (i) each  $Q_j^k$  has sides parallel to the coordinate axes and edges of length  $2^{-k}$ ,
- (ii) the interiors of the cubes  $Q_j^k$  and  $Q_i^k$  are mutually disjoint provided  $i \neq j$ ,
- (iii) the distance between the cube  $Q_j^k$  and  $E$  satisfies the following inequalities

$$(2.3) \quad 2^{-k} \sqrt{n} \leq d(Q_j^k, E) \leq 2^{2-k} \sqrt{n}.$$

The decomposition (2.2) is called the *Whitney decomposition*, the cubes  $Q_j^k$  are called the *Whitney cubes* and the set  $\{Q_j^k: j = 1, \dots, N_k\}$  is called the  $k^{\text{th}}$  *generation* of the cubes.

For  $0 < s < t < \infty$  and  $E \subset \mathbb{R}^n$ , we define

$$E(s) = \{x \in \mathbb{R}^n: d(E, x) \leq s\}$$

and

$$(2.4) \quad E(s, t) = \{x \in \mathbb{R}^n: s \leq d(E, x) \leq t\}.$$

Because  $\text{diam}(Q_j^k) = \sqrt{n}2^{-k}$  we have by (2.3)

$$(2.5) \quad Q_j^k \subset E(\sqrt{n}2^{-k}, 5\sqrt{n}2^{-k}).$$

We say that  $E \subset \mathbb{R}^n$  is  $(\alpha, r_0)$ -uniformly porous [Va2] if for all  $x \in E$  and  $0 < r < r_0$  there is  $y \in B(x, r)$  with  $d(y, E) \geq \alpha r$ . A set  $E$  is uniformly porous, if it is  $(\alpha, r_0)$ -uniformly porous for some  $\alpha, r_0 > 0$ . We refer to  $\alpha, r_0$  and  $n$  as the *uniform porosity data*.

A set  $E \subset \mathbb{R}^n$  is  $Q$ -regular for  $0 < Q < n$ , if there is a (Borel regular, outer-) measure  $\mu$  with  $\text{spt}(\mu) = E$  and constants  $0 < \alpha \leq \beta < \infty$  such that

$$\alpha r^Q \leq \mu(B(x, r)) \leq \beta r^Q, \text{ for all } x \in E \text{ and } 0 < r < \text{diam}(E).$$

Here  $\text{spt}(\mu)$  denotes the smallest closed set with full  $\mu$ -measure. We refer to  $Q, \alpha, \beta$  and  $\text{diam}(E)$  as the  *$Q$ -regularity data*.

There is a close connection between the notions of uniform porosity and  $Q$ -regularity. Indeed, it is well known and easy to see (e.g. [BHR, Lemma 3.12]) that if  $E$  is  $Q$ -regular for some  $0 < Q < n$ , then it is uniformly porous. Conversely, if  $E$  is uniformly porous, then it is a subset of some  $Q$ -regular set for

some  $0 < Q < n$ . See [JKRRS, Theorem 5.3] for a more precise quantitative statement.

### 3. NUMBER OF WHITNEY CUBES

Recall that  $N_k$  denotes the number of the  $k^{\text{th}}$  generation Whitney cubes of  $\mathbb{R}^n \setminus E$ .

**Lemma 3.1.** *If  $E \subset \mathbb{R}^n$  is compact and uniformly porous, then there are constants  $0 < c < C < \infty$  and  $k_0 \in \mathbb{N}$  (depending only on the uniform porosity data) such that for  $k \geq k_0$ , it holds*

$$c2^{kn}m(E(2^{-k})) \leq \tilde{N}_k \leq C2^{kn}m(E(2^{-k})),$$

where  $\tilde{N}_k = N_{k+n_0} + \dots + N_{k+n_1}$ ,  $n_0$  is the smallest integer satisfying  $8\sqrt{n}2^{-n_0} \leq 1$  and  $n_1$  is the largest integer with  $80\sqrt{n}2^{-n_1} \geq 1$ .

*Proof.* Let  $0 < \lambda < 1 < \Lambda < \infty$ . Choose  $0 < \alpha < (1 + \lambda)/(2\Lambda)$  and  $r_0 > 0$  such that  $E$  is  $(2\alpha, r_0)$ -uniformly porous. Given  $r > 0$  such that  $R = \frac{1}{2\alpha}(1 + \lambda)r < r_0$ , let  $B_1, \dots, B_N$  be a maximal collection of disjoint balls of radius  $R$  centered at  $E$ . Then by elementary geometry

$$(3.2) \quad \Omega_n NR^n \leq m(E(R)) \leq \Omega_n 3^n NR^n.$$

For each  $B_i$ , we can find  $y_i \in \frac{1}{2}B_i$  with  $d(y_i, E) \geq \alpha R = \frac{1}{2}(1 + \lambda)r$ . Moreover (since we live in the Euclidean space), we can in fact find such  $y_i$  with  $d(y_i, E) = \frac{1}{2}(1 + \lambda)r$ . Then  $B(y_i, \frac{1}{2}(1 - \lambda)r) \subset E(r) \setminus E(\lambda r)$  and thus, combined with (3.2),

$$\begin{aligned} m(E(r) \setminus E(\lambda r)) &\geq N\Omega_n \left(\frac{1 - \lambda}{2}\right)^n r^n \geq \left(\frac{\alpha(1 - \lambda)}{3(1 + \lambda)}\right)^n 3^n \Omega_n NR^n \\ &\geq \left(\frac{\alpha(1 - \lambda)}{3(1 + \lambda)}\right)^n m(E(\Lambda r)), \end{aligned}$$

recall that  $R \geq \Lambda r$ .

Applying the above estimate with  $r = 2^{-k-3}$ ,  $\lambda = \frac{1}{2}$  and  $\Lambda = 8$  and using (2.5) implies

$$\begin{aligned} m(E(2^{-k})) &\leq cm(E(2^{-k-3}) \setminus E(2^{-k-4})) \\ &\leq c \sum_{j=n_0}^{n_1} N_{k+j} 2^{-(k+j)n} \leq cm(E(2^{-k})) \end{aligned}$$

where  $c < +\infty$  depends only on  $n$  and  $\alpha$ . This completes the proof.  $\square$

We also need the following simple fact

**Lemma 3.3.** *If  $E \subset \mathbb{R}^n$  is compact and  $Q$ -regular for some  $0 < Q < n$ , then there are constants  $0 < c < C < \infty$  (depending only on the  $Q$ -regularity data) such that  $cr^{n-Q} \leq m(E(r)) \leq Cr^{n-Q}$  for  $0 < r < \text{diam}(E)$ .*

*Proof.* Let  $B_1, \dots, B_N$  be a maximal collection of balls of radius  $r$  centered at  $E$ . Then, as in (3.2),

$$\bigcup_{i=1}^N B_i \subset E(r) \subset \bigcup_{i=1}^N 3B_i,$$

and in particular,

$$(3.4) \quad N\Omega_n r^n \leq m(E(r)) \leq N\Omega_n 3^n r^n.$$

Using the  $Q$ -regularity, we have

$$(3.5) \quad N\alpha r^Q \leq \sum_i \mu(B_i) \leq \mu(E) \leq \sum_i \mu(3B_i) \leq N\beta 3^Q r^Q.$$

The claim follows by combining (3.4) and (3.5).  $\square$

Putting the two previous Lemmas together, we obtain:

**Corollary 3.6.** *If  $E \subset \mathbb{R}^n$  is compact and  $Q$ -regular for some  $0 < Q < n$ , then there are constants  $0 < c < C < \infty$  and  $k_0 \in \mathbb{N}$  (depending only on the  $Q$ -regularity data) such that for all  $k \geq k_0$ , it holds*

$$c2^{kQ} \leq \tilde{N}_k \leq C2^{kQ}.$$

**Remark 3.7.** The porosity and regularity assumptions in the above results are essential because without them, we cannot guarantee that  $m(E(2r) \setminus E(r))$  is comparable to  $m(E(r))$  for small  $r > 0$ .

#### 4. GROWTH OF THE QUASIHYPHERBOLIC VOLUME

As mentioned in the Introduction, our goal is to find estimates for the asymptotic behaviour of  $\text{vol}_k(B_k(x, r))$  as  $r \rightarrow \infty$  for  $x \in G$  when the domain  $G$  is fixed. As an example, we first consider very regular domains for which the correct asymptotics can be obtained by direct calculation.

**4.1. Unit ball.** Let us find the quasihyperbolic volume of  $B_k(r)$  in the unit ball  $G = B^n$ . We need the following integral representation for the hypergeometric function: for  $c > b > 0$ ,

$$F(a, b; c; x) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a} dt$$

where  $B(b, c - b) = \Gamma(b)\Gamma(c - b)/\Gamma(c)$ . By definition

$$\begin{aligned}
 \text{vol}_k(B^n(r)) &= \int_{B^n(r)} \frac{dm(x)}{(1 - |x|)^n} = \omega_{n-1} \int_0^r \frac{t^{n-1}}{(1 - t)^n} dt \\
 &= \omega_{n-1} r^n \int_0^1 t^{n-1} (1 - rt)^{-n} dt \\
 &= \frac{\omega_{n-1} r^n}{n} F(n, n; n + 1; r) \\
 (4.2) \quad &= \frac{\omega_{n-1} r^n}{n(1 - r)^{n-1}} F(1, 1; n + 1; r),
 \end{aligned}$$

where the last equality follows from [AS, 15.3.3]

$$F(a, b; c; x) = (1 - x)^{c-a-b} F(c - a, c - b; c; x).$$

Hence for  $0 < r < \infty$

$$\begin{aligned}
 \text{vol}_k(B_k(r)) &= \frac{\omega_{n-1} (\tanh(r/2))^n}{n (1 - \tanh(r/2))^{n-1}} F(1, 1; n + 1; \tanh(r/2)) \\
 &\sim \frac{2^{1-n} \omega_{n-1}}{n - 1} e^{(n-1)r} \quad (r \rightarrow \infty),
 \end{aligned}$$

since by [AS, 15.1.20]

$$F(1, 1; n + 1; 1) = \frac{\Gamma(n + 1)\Gamma(n - 1)}{\Gamma(n)^2} = \frac{n}{n - 1}.$$

**Corollary 4.3.** *For  $s > 0$ ,  $\lambda > 1$ ,  $G = B^n$  and  $E = \partial G$*

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \rightarrow 1 - \lambda^{1-n}$$

as  $s \rightarrow 0$ .

*Proof.* Let  $0 < s < 1/\lambda$ . Then the claim is equivalent to

$$\frac{\text{vol}_k(B^n(1 - s) \setminus B^n(1 - \lambda s))}{\text{vol}_k(B^n(1 - s))} \rightarrow 1 - \lambda^{1-n}$$

as  $s \rightarrow 0$ .

By (4.2)

$$\begin{aligned}
 &\frac{\text{vol}_k(B^n(1 - s) \setminus B^n(1 - \lambda s))}{\text{vol}_k(B^n(1 - s))} \\
 &= 1 - \frac{\lambda^{1-n} (1 - \lambda s)^n F(1, 1; n + 1; 1 - \lambda s)}{(1 - s)^n F(1, 1; n + 1; 1 - s)} \\
 &\rightarrow 1 - \lambda^{1-n} \quad (s \rightarrow 0).
 \end{aligned}$$

□

**Remark 4.4.** For  $G = B^n$ , let  $m_h(B^n(r))$  be the hyperbolic volume of the Euclidean ball  $B^n(r)$ . Then we have

$$\begin{aligned} \left(\frac{2}{1+r}\right)^n \text{vol}_k(B^n(r)) &= \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1+r)^n (1-t)^n} dt \\ &\leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t^2)^n} dt = m_h(B^n(r)) \end{aligned}$$

and

$$m_h(B^n(r)) \leq \omega_{n-1} \int_0^r \frac{2^n t^{n-1}}{(1-t)^n} dt = 2^n \text{vol}_k(B^n(r)).$$

It is easy to see that  $m_h(B^n(r)) = \omega_{n-1} 2^n r^n F(n/2, n; 1+n/2; r^2)/n$ . Corollary 4.3 is true for the hyperbolic metric with the same limiting value  $1 - \lambda^{n-1}$ .

**4.5. Punctured space.** We will consider the growth of quasihyperbolic volume in multiply connected domains. It can be shown that in  $\mathbb{R}^2 \setminus \{0\}$  the quasihyperbolic area of  $B_k(x, r)$  is equal to  $\pi r^2$ , if  $r \leq \pi$ , and  $2\pi\sqrt{r^2 - \pi^2} + 2\pi^2 \arctan(\pi/\sqrt{r^2 - \pi^2})$ , if  $r > \pi$ . Therefore

$$2\pi\sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x, r)) \leq 4\pi r$$

if  $r > \pi$ . We will now find similar lower and upper bounds for the quasihyperbolic volume of  $B_k(x, r)$  in  $\mathbb{R}^n \setminus \{0\}$  for  $n > 2$ .

**Proposition 4.6.** For  $x \in \mathbb{R}^n \setminus \{0\}$ ,  $n > 2$  and  $r > \pi$

$$2\omega_{n-1}\sqrt{r^2 - \pi^2} \leq \text{vol}_k(B_k(x, r)) \leq 2\omega_{n-1}r.$$

*Proof.* Since  $B_k(x, r)$  is invariant in the inversion in  $S^{n-1}(|x|)$  we have  $\text{vol}_k(B_k(x, r)) = 2\text{vol}_k(B_k(x, r) \cap B^n(|x|))$ . By [V1, (3.9)]  $B_k(x, r) \subset \mathbb{R}^n \setminus B^n(|x|e^{-r})$ . Let  $S = S^{n-1}(|x|e^{-\sqrt{r^2 - \pi^2}})$ , then by (2.1)  $\max_{y \in S} k_{\mathbb{R}^n \setminus \{0\}}(x, y) = r$  implies  $B^n(|x|) \setminus B^n(|x|e^{-\sqrt{r^2 - \pi^2}}) \subset B_k(x, r)$ . Hence we have

$$(4.7) \quad \text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-\sqrt{r^2 - \pi^2}})) \leq \text{vol}_k(B_k(x, r) \cap B^n(|x|))$$

and

$$(4.8) \quad \text{vol}_k(B_k(x, r) \cap B^n(|x|)) \leq \text{vol}_k(B^n(|x|) \setminus B^n(|x|e^{-r})).$$

Next let us find the quasihyperbolic volume of annulus  $E(a, b) = \{x \in \mathbb{R}^n \mid a < |x| < b\}$ ,  $0 < a < b < \infty$ . By definition (4.9)

$$\text{vol}_k(E(a, b)) = \int_{E(a, b)} \frac{dm(x)}{|x|^n} = \omega_{n-1} \int_a^b \frac{t^{n-1}}{t^n} dt = \omega_{n-1} \log \frac{b}{a}.$$

Now the assertion follows from the equations (4.7) and (4.8).  $\square$

**Corollary 4.10.** *For  $s > 0$ ,  $\lambda > 1$ ,  $G = B^n \setminus \{0\}$  and  $E = \partial G$*

$$\frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} \rightarrow 1 - \lambda^{1-n}$$

as  $s \rightarrow 0$ .

*Proof.* Let  $s < 1/\lambda$ . Then  $E(s, \lambda s) = E_1 \cup E_2$  where  $E_1 = B^n(1-s) \setminus B^n(1-\lambda s)$  and  $E_2 = B^n(\lambda s) \setminus B^n(s)$ . Then by (4.2) and (4.9)

$$\begin{aligned} \text{vol}_{k_G}(E(s, \lambda s)) &= \text{vol}_{k_{B^n}}(E_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E_2) \\ &= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n+1, 1-s) \\ &\quad - \frac{\omega_{n-1}(1-\lambda s)^n}{n(\lambda s)^{n-1}} F(1, 1; n+1, 1-\lambda s) + \omega_{n-1} \log \lambda \\ &\sim \frac{\omega_{n-1}}{n-1} (1 - \lambda^{1-n}) s^{1-n} \quad (s \rightarrow 0). \end{aligned}$$

Similarly  $E(s, \infty) = B^n(1-s) \setminus B^n(s) = E'_1 \cup E'_2$  where  $E'_1 = B^n(1-s) \setminus B^n(1/2)$  and  $E'_2 = B^n(1/2) \setminus B^n(s)$ , and

$$\begin{aligned} \text{vol}_{k_G}(E(s, \infty)) &= \text{vol}_{k_{B^n}}(E'_1) + \text{vol}_{k_{\mathbb{R}^n \setminus \{0\}}}(E'_2) \\ &= \frac{\omega_{n-1}(1-s)^n}{ns^{n-1}} F(1, 1; n+1, 1-s) \\ &\quad - \frac{\omega_{n-1}}{2n} F(1, 1; n+1, 1/2) + \omega_{n-1} \log(1/2s) \\ &\sim \frac{\omega_{n-1}}{n-1} s^{1-n} + \omega_{n-1} \log(1/s) \quad (s \rightarrow 0). \end{aligned}$$

Now we have

$$\lim_{s \rightarrow 0} \frac{\text{vol}_k(E(s, \lambda s))}{\text{vol}_k(E(s, \infty))} = \lim_{s \rightarrow 0} \frac{\frac{\omega_{n-1}}{n-1} (1 - \lambda^{1-n}) s^{1-n}}{\frac{\omega_{n-1}}{n-1} s^{1-n} + \omega_{n-1} \log(1/s)} = 1 - \lambda^{1-n}.$$

□

**4.11. Half space.** The next two propositions concern the case of the half space, in which case the quasihyperbolic volume coincides with the classical hyperbolic volume. The hyperbolic volume of hyperbolic simplexes in the upper half space has been considered in [Mi].

**Proposition 4.12.** *For  $x \in \mathbb{H}^2$  and  $r > 0$  we have*

$$\text{vol}_k(B_k(x, r)) = 2\pi(\cosh r - 1).$$

*Proof.* Let us choose  $x = te_2$  implying that

$$B_k(x, r) = B^2(e_2 \cosh r, \sinh r).$$



Now

$$\begin{aligned}\text{vol}_k(B_k(x, r)) &= \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \frac{\sqrt{t^2 \sinh^2 r - (\cosh r - h)^2}}{h^2} dh \\ &= 2\pi(\cosh r - 1).\end{aligned}$$

□

**Proposition 4.13.** *For  $x \in \mathbb{H}^3$  and  $r > 0$  we have*

$$\text{vol}_k(B_k(x, r)) = \pi(\sinh(2r) - 2r).$$

*Proof.* Let us choose  $x = te_2$  implying that

$$B_k(x, r) = B^2(e_2 \cosh r, \sinh r).$$

Now

$$\begin{aligned}\text{vol}_k(B_k(x, r)) &= \int_{t \cosh r - t \sinh r}^{t \cosh r + t \sinh r} \frac{\pi(t^2 \sinh^2 r - (\cosh r - h)^2)}{h^3} dh \\ &= \pi(\sinh(2r) - 2r).\end{aligned}$$

□

**4.14. More general domains.** Our next goal is to prove growth estimates of the above type for larger class of domains, so called  $\varphi$ -uniform domains. To that end, we first observe that the quasi-hyperbolic volume of the Whitney cubes is essentially constant.

**Lemma 4.15.** *Let  $E$  be a closed subset of  $\mathbb{R}^n$  and let  $G$  be a component of  $\mathbb{R}^n \setminus E$ . Then for the Whitney cubes of  $\mathbb{R}^n \setminus E$  contained in  $G$  we have*

$$2^{-2n}n^{-n/2} \leq \text{vol}_k(Q_j^k) \leq n^{-n/2}.$$

*Proof.* Since  $m(Q_j^k) = 2^{-kn}$  we have by (2.3)

$$\text{vol}_k(Q_j^k) \geq \frac{2^{(k-2)n}}{\sqrt{n}} 2^{-kn} = 2^{-2n}n^{-n/2},$$

and

$$\text{vol}_k(Q_j^k) \leq \frac{2^{kn}}{\sqrt{n}} 2^{-kn} = n^{-n/2}.$$

□

**Theorem 4.16.** *Let  $G$  be a proper subdomain of  $\mathbb{R}^n$  with compact and  $Q$ -regular ( $0 < Q < n$ ) boundary. There exists a constant  $C < \infty$  such that for each  $x \in G$  and sufficiently large  $r > 0$ , we have*

$$\text{vol}_k(B_k(x, r)) \leq Ce^{Qr}.$$

*Proof.* Let  $E = \partial G$ . We have

$$B_k(x, r) \subset \{z \in G : e^{-r}d(x) < d(z) < e^r d(x)\} =: D$$

and

$$Q_j^k \subset E(2^{-k}\sqrt{n}, 5 \cdot 2^{-k}\sqrt{n}).$$

If  $5 \cdot 2^{-k}\sqrt{n} < e^{-r}d(x)$  or  $2^{-k}\sqrt{n} > e^r d(x)$ , i.e.

$$k > \frac{r + \log(5\sqrt{n}/d(x))}{\log 2} =: K, \quad \text{or} \quad k < -\frac{r - \log(\sqrt{n}/d(x))}{\log 2} =: K'$$

then  $Q_j^k \cap D = \emptyset$ , and thus

$$B_k(x, r) \subset \bigcup_{k=[K']}^{[K]} \bigcup_{j=1}^{N_k} Q_j^k.$$

Note that  $N_k \leq M(\text{diam}(E))2^{nk}$  always holds for all  $k \in \mathbb{N}$  [MV] and therefore  $N_k \leq C(\text{diam}(E), k_0)$  for  $k \leq k_0$ , where  $k_0$  is as in Lemma 3.1. Hence for all sufficiently large  $r$ , we may estimate

$$\begin{aligned} \text{vol}_k(B_k(x, r)) &\leq \sum_{k=[K']}^{[K]} \sum_{j=1}^{N_k} \text{vol}_k(Q_j^k) \\ &\leq n^{-n/2} \sum_{k=[K']}^{[K]} N_k \\ &\leq n^{-n/2} (k_0 - [K']) C(\text{diam}(E), k_0) \\ &\quad + n^{-n/2} \sum_{k=k_0}^{[K]} c 2^{kn} m(E(2^{-k})) \\ &\leq n^{-n/2} (k_0 - [K']) C(\text{diam}(E), k_0) \\ &\quad + n^{-n/2} \sum_{k=k_0}^{[K]} c 2^{kn} c' (2^{-k})^{n-Q} \\ &\leq n^{-n/2} (k_0 - [K']) C(\text{diam}(E), k_0) \\ &\quad + \frac{cc' n^{-n/2}}{1 - 2^{-Q}} 2^{KQ} \\ &\leq C e^{Qr}, \end{aligned}$$

where after the first inequality we only sum over those Whitney cubes that are contained in  $G$ , the third and fourth inequalities follow from Lemmas 3.1 and 3.3, respectively, and the last inequality holds for sufficiently large  $r$  by the definition of  $K$  and  $K'$ . Note that by adapting the constants, we may assume that Lemma 3.3 holds for all  $0 < r < 2^{-k_0}$ .  $\square$

Let  $\varphi : [0, \infty) \rightarrow [0, \infty)$  be a continuous strictly increasing function with  $\varphi(0) = 0$ . A domain  $G \subset \mathbb{R}^n$  is said to be  $\varphi$ -uniform

[V2] if

$$(4.17) \quad k_G(x, y) \leq \varphi(|x - y| / \min\{d(x, \partial G), d(y, \partial G)\})$$

for all  $x, y \in G$ . In particular, the domain  $G$  is  $C$ -uniform if  $\varphi(t) = C \log(1 + t)$  with  $C > 1$ . A domain is called uniform if it is a  $C$ -uniform domain for some  $C > 1$ . These domains have been studied by many authors, see [GO, KLSV, M, Va3, V2].

**Lemma 4.18.** *Let  $\emptyset \neq E \subset \mathbb{R}^n$  be an  $(\alpha, r_0)$ -uniformly porous compact set such that  $G = \mathbb{R}^n \setminus E$  is a  $\psi$ -uniform domain and let  $x \in G$ . Then there exists  $r_1 = r_1(x) > 0$  such that*

$$d(w, B_k(x, r)) \leq \frac{\text{diam}(\partial G) + d(x) + r_0}{\alpha \psi^{-1}(r)}$$

for all  $w \in \partial G$  and  $r \geq r_1$ .

*Proof.* Let  $w \in \partial G$ . Since  $\partial G$  is uniformly  $(\alpha, r_0)$ -porous, for each  $s \in (0, r_0]$  there exists  $y \in G \cap B^n(w, s)$  with  $d(y) > \alpha s$ . By  $\psi$ -uniformity of  $G$ ,

$$k_G(x, y) \leq \psi \left( \frac{|x - y|}{\min\{d(x), d(y)\}} \right) \leq \psi \left( \frac{\text{diam}(\partial G) + d(x) + r_0}{\min\{d(x), \alpha s\}} \right),$$

which implies

$$d \left( w, B_k \left( x, \psi \left( \frac{\text{diam}(\partial G) + d(x) + r_0}{\min\{d(x), \alpha s\}} \right) \right) \right) \leq s.$$

Substituting  $s = (\text{diam}(\partial G) + d(x) + r_0) / (\alpha \psi^{-1}(r))$  this reads

$$d(w, B_k(x, r)) \leq \frac{\text{diam}(\partial G) + d(x) + r_0}{\alpha \psi^{-1}(r)}$$

for  $r \geq \psi((\text{diam}(\partial G) + d(x) + r_0) / \min\{d(x), \alpha r_0\}) =: r_1$ .  $\square$

**Theorem 4.19.** *Let  $E \subset \mathbb{R}^n$  be a closed  $Q$ -regular set with  $0 < Q < n$  such that  $G = \mathbb{R}^n \setminus E$  is a  $\psi$ -uniform domain. Then for each  $x \in D$  there is  $c > 0$  and  $r_1 > 0$  such that*

$$\text{vol}_k(B_k(x, r)) \geq c(\psi^{-1}(r))^Q,$$

for all  $r > r_1$ .

*Proof.* Let  $\alpha, r_0 > 0$  be such that  $\partial G$  is  $(\alpha, r_0)$ -uniformly porous and set  $C = (\text{diam}(\partial G) + d(x) + r_0) / (\alpha \sqrt{n})$ . Furthermore, let  $K$  be the largest integer such that

$$2^{-K} \geq \frac{C}{\psi^{-1}(r)}.$$

Then, by combining Lemma 4.18 and (2.5), we see that for  $m \leq K$ , all the  $m^{\text{th}}$ -generation Whitney cubes belong to  $B_k(x, r)$ . Together with Corollary 3.6 and 4.15 this yields for large values of

$r$  the required estimate

$$\mathrm{vol}_k(B_k(x, r)) \geq c_0 \tilde{N}_K \geq c_1 2^{KQ} \geq c(\psi^{-1}(r))^Q,$$

where  $c > 0$  only depends on the  $Q$ -regularity data and  $d(x)$ .  $\square$

By combining Theorem 4.16, Theorem 4.19 and the definition of uniform domain we get the following corollary.

**Corollary 4.20.** *Let  $E \subset \mathbb{R}^n$  be a compact  $Q$ -regular set with  $0 < Q < n$  such that  $G = \mathbb{R}^n \setminus E$  is a uniform domain with the uniformity constant  $L > 1$ . Then for each  $x \in G$  and sufficiently large  $r > 0$ ,*

$$ce^{Qr/L} \leq \mathrm{vol}_k(B_k(x, r)) \leq Ce^{Qr},$$

where  $C < \infty$  only depends on the  $Q$ -regularity data and  $L$  and  $c > 0$  depends only on the  $Q$ -regularity data,  $L$ , and  $d(x)$ .

**Example 4.21.** Let  $E \subset \mathbb{R}^n$  be a self-similar set whose complement is a uniform domain. For instance,  $E$  can be a Cantor set on a hyperplane, the  $\frac{1}{4}$ -Cantor set in the plane, or more generally any self-similar set satisfying the strong separation condition (See e.g. [F] for the definitions). Then  $E$  is  $Q$ -regular in its dimension and Corollary 4.20 can be applied.

**Remark 4.22.** Although we stated the Theorem 4.19 for unbounded domains, essentially the same proof works for all  $\psi$ -uniform domains  $G \subset \mathbb{R}^n$  whose boundary is  $Q$ -regular and uniformly porous in  $G$ . The last assumption means that the porosity holes in the definition of the uniform porosity lie completely inside  $G$ . For instance, in  $\mathbb{R}^2$ ,  $\partial G$  can be the Von-Koch snowflake curve. More generally, the boundary could be the union of a finite number of such  $Q_i$ -regular pieces, and in this case we could apply the lower bound with the exponent  $r \min\{Q_i\}/L$  and the upper bound with the exponent  $r \max\{Q_i\}$ .

It is not known what is the best possible lower bound in Corollary 4.20. It is an interesting open problem if the exponent  $Qr/L$  could be replaced by  $cr$  for some uniform constant  $c > 0$  independent of  $L$ .

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